Spatiotemporal organization of coupled nonlinear pendula through impurities

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We study the effect of impurities introduced into a lattice and their ability to control the dynamical behavior of arrays of coupled nonlinear chaotic oscillators. In particular we show that under certain conditions a single impurity can produce simple spatiotemporal patterns in place of complex chaotic behavior for very long chains of oscillators. Under the same conditions we also examine the effect of disorder in the lengths of the pendula and explain the observed patterns and periodic behavior to be the result of inadvertent impurities.
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I. INTRODUCTION

In the past few years there has been a large number of studies focused on coupled nonlinear oscillators and their properties. The most widely used model is the damped Frenkel-Kontorova model in which a collection of oscillators with a sine nonlinearity are diffusively coupled to their nearest neighbors [1]. The relevance of such a system of equations to condensed matter systems, such as charge-density waves, and in Josephson junctions is well known. In particular long Josephson junctions with an applied dc bias voltage are well described by a perturbed sine-Gordon equation [2]. Thus one dimensional parallel arrays of Josephson junctions when biased by an external current should have the phase difference between the two sides of the *n*th junction well described by the equations of coupled pendula driven by a constant torque.

The problem of mutual synchronization or equivalently the stability of the in-phase state has been examined by a number of investigators [3]. It was found that even though there exists a broad range of parameters in which the in-phase state is stable, nevertheless, there is also a broad parameter regime in which it is unstable. Recent investigations [4] have unveiled the unexpected result of significantly increasing the synchronization of an array of Josephson junctions by introducing disorder. Indeed by allowing the critical currents to be randomly distributed it was found that for certain realizations the array exhibited a sharp increase in the level of synchronization. A similar problem was investigated recently by Wiesenfeld *et al.* [5] in a series array of junctions biased with a constant current and subject to a load. They showed that as a function of disorder in the natural frequencies, the array first undergoes a transition to partial synchronization followed by another that corresponds to complete phase locking. Other studies in this area include the effect of natural frequency distribution on cluster synchronization in oscillator arrays [6], the collective behavior of limit-cycle oscillators [7], the influence of quenched disorder on a coupled map model of earthquakes [8], investigations on the issue of optimal disorder for taming spatiotemporal chaos [9], and controlling localized spatiotemporal chaos in coupled map lattices [10].

In this paper we investigate the effect of disorder in the driven Frenkel-Kontorova model or equivalently in the diffusively coupled Josephson junctions in which the applied current at each junction is modulated by a common frequency. Because of the additional degree of freedom, driven oscillators can exhibit chaotic behavior and therefore the difficulty of synchronization increases considerably. This problem has been investigated recently by Braiman *et al.* [11] for the case of a one-dimensional chaotic array of forced damped nonlinear pendula, who have observed that by introducing a certain amount of disorder into the lengths of the pendula, complex but frequency locked spatiotemporal patterns can emerge in which the chaotic behavior is completely suppressed. We will investigate the same phenomenon but from a different viewpoint. Namely to what extent and for how large an array may we induce synchronization, given that all the pendula are chaotic except a single impurity introduced at a particular site of the array.

In the next sections we quickly outline the model that we will use to investigate the possibility of self-organization of large arrays of oscillators. In Sec. III we present calculations and numerical results and discuss them in detail. Finally in Sec. IV we present a summary of this work and our conclusions.

II. MODEL

To demonstrate our ideas more clearly we will use the model examined in Ref. [11],

$$ml_n^2\ddot{\theta}_n+\gamma\dot{\theta}_n=-mgl_n\sin\,\theta_n+\tau'+\tau\sin\,\omega t+k(\theta_{n+1}-2\theta_n+\theta_{n-1}), \tag{1}$$

where $n=1,2,...,N$ and we will assume free boundary conditions, i.e., $\theta_0=\theta_1$, $\theta_N=\theta_{N+1}$. The parameters used in the calculations are the gravitational acceleration $g=1$, the mass

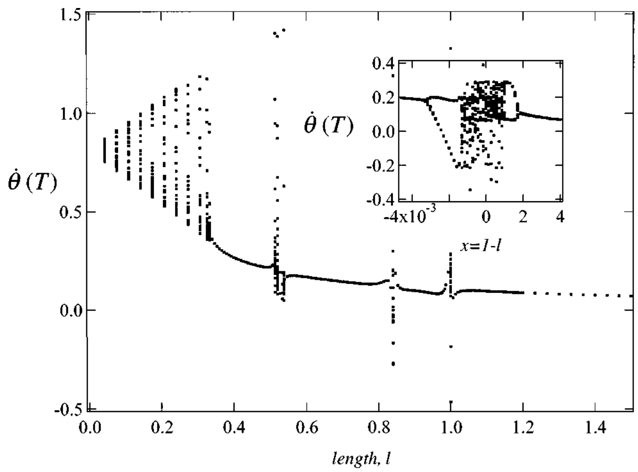

FIG. 1. The bifurcation diagram of the single isolated pendulum. The velocity $\dot{\theta}$ at each period of the driving frequency is plotted as a function of the length l. Inset shows an expanded version of the region about $l\sim1.0$ and the abscissa is defined by $x=1-l$.

of the pendulum $m=1$, the dc torque $\tau'=0.7155$, the ac torque $\tau=0.4$, the angular frequency $\omega=0.25$, and the damping $\gamma=0.75$. Thus at each site there is an underdamped oscillator with a pendulum length l_n. The parameter k denotes the coupling between neighboring pendula and its value is nominally taken to be 0.5 as in Ref. [11]. One finds that each isolated pendulum is chaotic for values of $l_n\sim1.0$ and it is characterized by a single positive Lyapunov exponent. For values larger than one, the pendulum executes a libration in which it oscillates about its equilibrium position without overturning, that is without the angle θ increasing past 2π. On the other hand for pendulum lengths shorter than one, the pendulum executes a rotation where the combined torques rotate the pendulum over the top and the angle θ past 2π.

III. CALCULATIONS AND DISCUSSION

It is instructive first to examine the bifurcation diagram as a function of the length of a single isolated pendulum and identify the possible chaotic regimes. All the calculations were done using a fourth order Runge-Kutta routine with a step size of $dt=0.001$. The defining parameter was the period of the driving signal and not the frequency. Thus a stroboscopic Poincaré cut was obtained without running into inaccurate representations because of accumulated error in the time definition at which the cut was done. Figure 1 shows the bifurcation diagram of the single isolated pendulum. The velocity $\dot{\theta}$ is plotted as a function of the length l at each period of the driving signal. We notice four regions in which the pendulum exhibits chaotic behavior. There are three narrow chaotic regimes at $l\sim1.0$, $l\sim0.84$, $l\sim0.52$, and a broad one for $l<0.35$. We further examine carefully the region around $l\sim1.0$, and find, as shown in the inset of Fig. 1, that the chaotic region has an extent of $0.998<l<1.002$, and it is indeed quite a narrow region. Similar narrow bands appear also for the other chaotic regimes but they are not shown here explicitly. This investigation then allows us to completely characterize whether a given pendulum in the chain is chaotic or not.

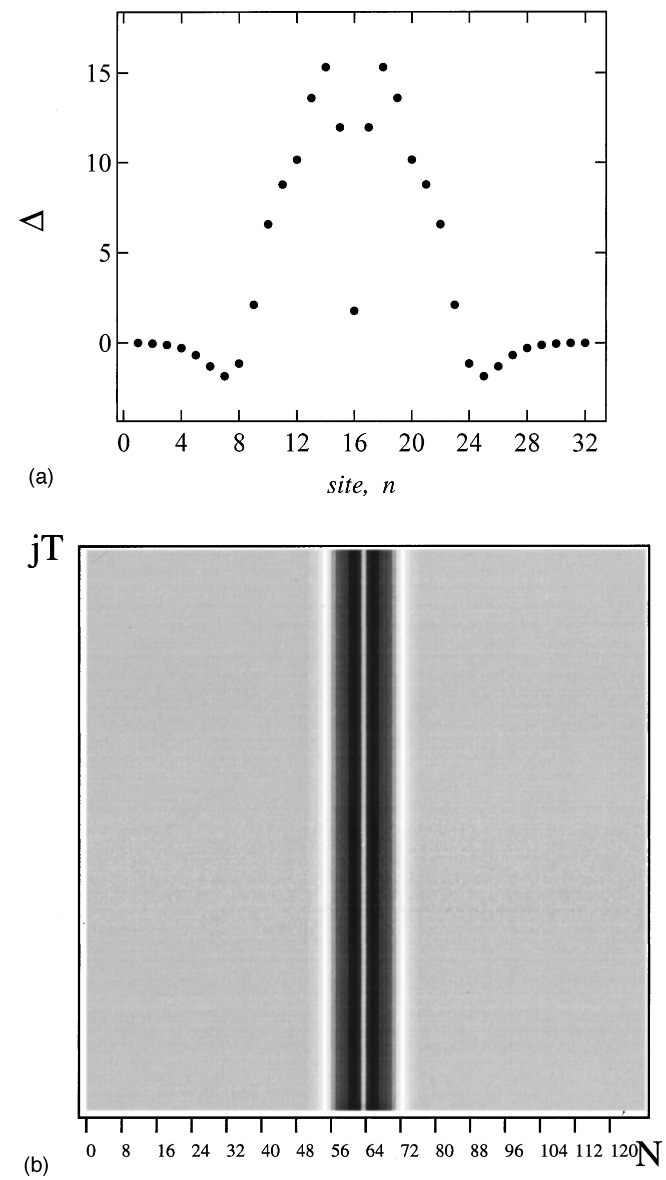

FIG. 2. (a) The difference in velocity of the nth pendulum with respect to the first one $\Delta\dot{\theta}_n=\dot{\theta}_n-\dot{\theta}_1$, as a function for $l_n=1.15$ and $l_{imp}=1.0$. The ordinate is defined as $[\Delta=(\dot{\theta}_n-\dot{\theta}_1)\times1000]$. (b) Gray scale evolution of $\dot{\theta}_n(T)$ for $N=128$, and $l_n=0.8$ and $l_{imp}=l_{64}=1.0$. The pattern is plotted for 50 periods of the drive ($j=0-50$).

It was argued in Ref. [11] that one of the possible mechanisms by which disorder may stabilize a chaotic array involves the removal of some of the oscillators from their chaotic band, thus creating distinctly different clusters of oscillators. These periodic populations then force a locking of the remaining chaotic clusters to the external drive and create periodic solutions to the equations of motion. Similar effects have been seen by other authors [12] for identical coupled Hénon maps in which at appropriate couplings clustering may occur. One then finds that there are clusters of periodic solutions of various periods interspersed with chaotic clusters.

This idea of small synchronized domains that have crystallized around nonchaotic impurities induced by disorder

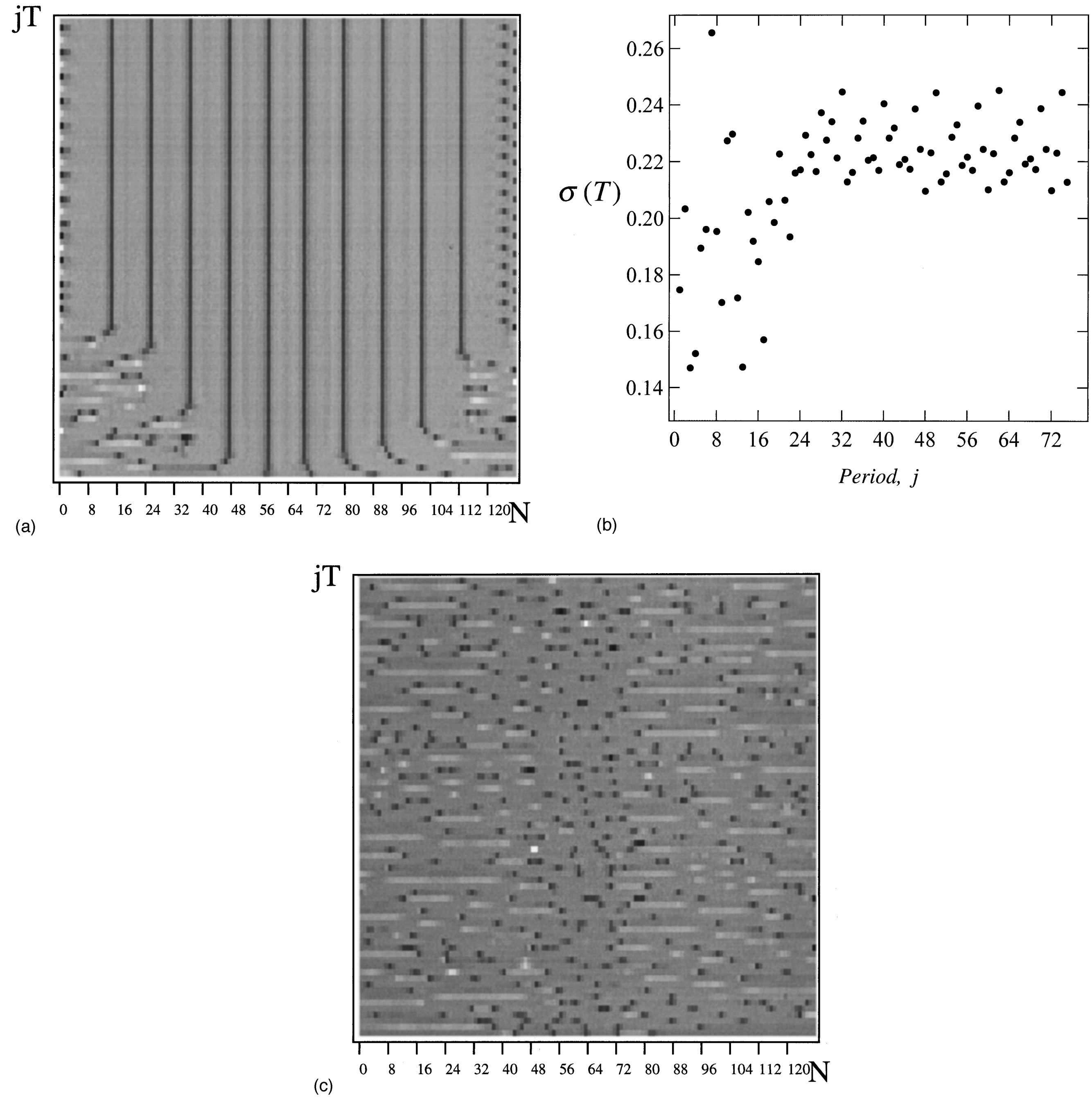

FIG. 3. (a) Spatiotemporal pattern produced by an $l_{\mathrm{imp}}=0.8$ impurity at site 64 in a 128 array of oscillators with $l_n=1.0$ and for 75 periods. (b) The measure $\sigma(T)$ for the configuration examined in (a) plotted as a function of period number j. (c) Chaotic pattern produced by an $l_{\mathrm{imp}}=1.2$ impurity at site 64 in a 128 array of oscillators with $l_n=1.0$. The pattern is plotted for 100 periods.

can be further extended. The underlying physical idea is borrowed from the physics of disordered solids and already used in some nonlinear studies [14], where it is known that defects create localized excitations in space around their positions [15]. As a result we will focus on the problem of what are the effects of a single impurity with length l_{imp}, at site M in a chain of equal-length pendula.

The simplest configuration we examine first is that of a single chaotic pendulum in a sea of identical nonchaotic pendula. Our numerical calculations indicate that in the absence of the chaotic impurity the solution for the chain is a uniform periodic solution with

$$\dot{\theta}_n(t)=\Phi(t) \tag{2}$$

independent of the site.

The inclusion of the chaotic impurity is to simply distort a small region around it in a symmetric manner in which to a large extent the rest of the lattice does not participate, however, maintaining frequency synchronization. In Fig. 2(a) we show a sample calculation of a chain of 32 oscillators having $l_n=1.15$ with an impurity of $l_{\mathrm{imp}}=1.0$ placed in the middle of the chain at site $n=16$. It plots the difference in velocity of the nth pendulum with respect to the first one $\Delta\dot{\theta}_n=\dot{\theta}_n$

$-\dot{\theta}_1$, as a function of site number. Around the defect the velocity difference is very small and symmetric. It broadens slightly and increases in value as the length difference increases. Nevertheless synchronization of the array is maintained as it is evident from a gray scale plot of the velocity as a function of site number shown in Fig. 2(b). In the vertical scale, time in integral multiples of the period of the drive are depicted. This calculation was performed for 128 pendula of length $l_n = 0.8$ and an impurity $l_{\rm imp} = 1.0$ at site 64. The darker shading indicates higher velocity. Thus the velocity of the impurity in this case is very close to that of the rest of the array with the highest velocity mismatch of couple sites away. Extensive numerical calculations with longer arrays have shown that the effects of the impurities are confined to its closest neighbors.

A more interesting avenue of investigation is the single nonchaotic impurity in a sea of identical chaotic pendula. For simplicity we restrict the length of the chaotic pendula to be 1.0 and investigate the effect of the length of a nonchaotic pendulum inserted at a convenient site, preferably in the middle of the chain, to the rest of the chain. Contrary to our expectations, it appears that for a large range of the length parameter of the impurity, a period one ($P1$) spatiotemporal pattern emerges. This organization of the chain was obtained for quite large chains. Most of the calculations were performed for chains of oscillators as large as 128 and for various lengths of the pendulum of the impurity. It appeared that short length impurities, in general, produced periodic patterns more often than impurities with large pendulum lengths for $k = 0.5$. This is quite evident in Fig. 3(a) where a spatiotemporal pattern is produced by an $l_{\rm imp} = 0.8$ impurity introduced at site 64 in a chain of 128 oscillators. A careful examination shows that after the transients have died out, approximately after 50 periods, the majority of the oscillators cluster into a $P1$ pattern except at the edges where it appears that the last three oscillators at each side belong to a $P12$ pattern.

A very convenient measure that allows a quick visualization of the average global spatiotemporal behavior of the chain and can ascertain its character, i.e., whether chaotic or periodic, and in addition identify the maximum period of the pattern, is the pattern average velocity:

$$\sigma(t) = \frac{1}{N} \sum_{n=1}^{N} \dot{\theta}_n(t). \tag{3}$$

This is one of a number of possible definitions and at this stage is the most useful and one of the quantities easiest to compute. This measure is computed at each period, and plotted as a function of jT, i.e., at each subsequent period. The measure $\sigma(jT)$ for the configuration examined in Fig. 3(a) is plotted in Fig. 3(b). Here the transients and the stabilization into a $P12$ cluster are immediately obvious. Modifying the boundary conditions from free to fixed $\theta_0 = \theta_{N+1} = 0$ leads to the formation of different patterns, emphasizing the importance of the edge effects. Moreover, changing to absorbing boundary conditions, i.e., the boundary oscillators are not driven, leads to complete elimination of the edge effects for this range of parameters. In fact it was found that in some cases absorbing boundary conditions were themselves quite sufficient to induce organization without the help of an impurity. In contrast, in Fig. 3(c) we show a chaotic pattern obtained by simply changing the length of the impurity to $l_{\rm imp} = 1.2$.

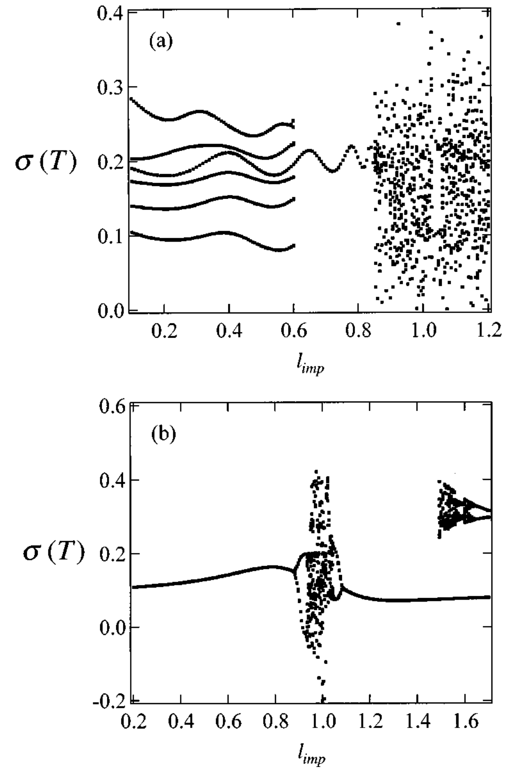

FIG. 4. (a) Bifurcation diagram of $\sigma(T)$ as a function $l_{\rm imp}$ for $k = 0.5$, and $N = 32$. (b) Bifurcation diagram of $\sigma(T)$ as a function $l_{\rm imp}$ for $k = 5.0$, and $N = 32$.

In view of these observations we infer the following plausible explanation. We recognize as in Refs. [12,13] that for a collection of coupled maps or coupled oscillators there are three important regimes of coupling strength to be distinguished. For no coupling or for very weak coupling there are N positive Lyapunov exponents, encompassing what Ref. [12] denotes as regime 1. In regime 2 of intermediate coupling strength there exists an extreme sensitivity to initial conditions and parameters. The number of positive Lyapunov exponents fluctuates wildly even for extremely small changes of the coupling. Finally in regime 3 the system possesses a single positive Lyapunov exponent. We think that in this coupling regime where there is only one positive Lyapunov exponent the insertion of a single nonchaotic oscillator could effectively modify the system's attractor and thus produce spatiotemporal organization.

If we repeat the same calculation but use instead $l_{\rm imp} = 0.52$ for the length of the impurity we still obtain a $P1$ pattern formation, if one excludes edge effects, which tend to disappear after quite long transients. We note here that apart from the chaotic regime around $l = 1.0$ that exists for the isolated pendulum, there are also other chaotic regimes as discussed previously. In particular if we use for the length of the impurity $l_{\rm imp} = 0.52$ then the isolated pendulum exhibits chaotic dynamics. Nevertheless, there is clear self-organization. This seems to support our previous hypothesis and render our explanation plausible.

To obtain a more global bifurcation diagram of the effects of the impurity as a function of the pendulum length at fixed

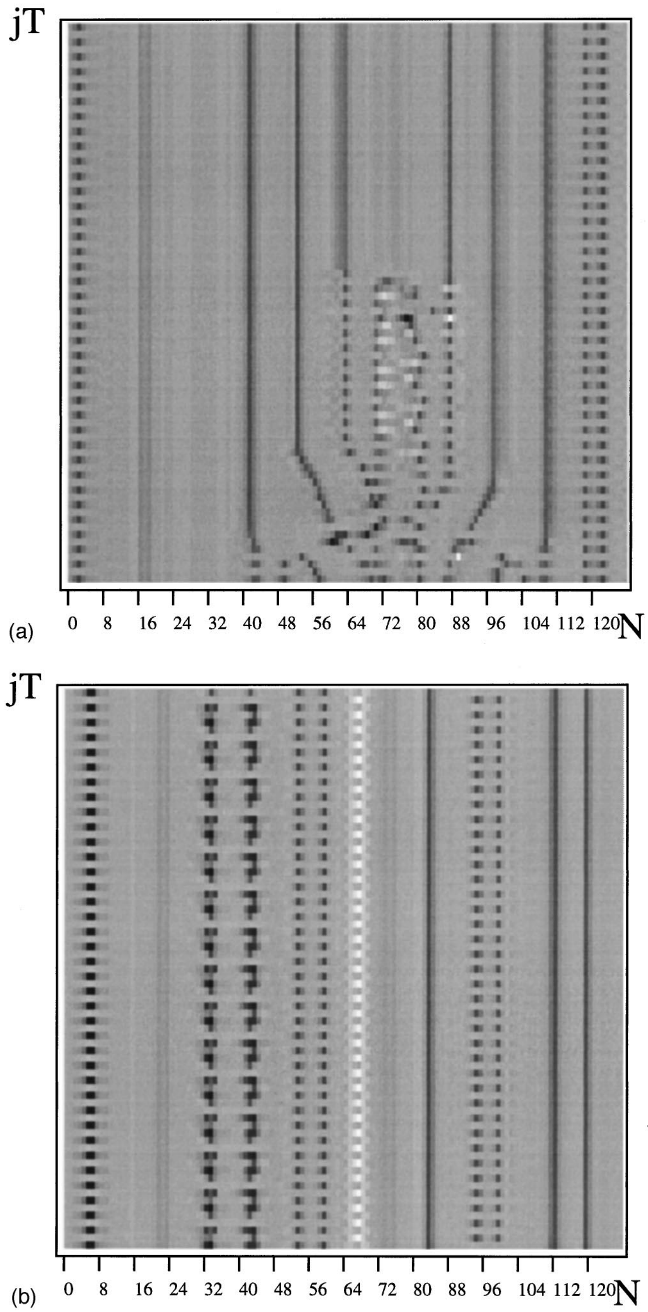

FIG. 5. (a) Uniform disorder of the pendula lengths in the interval [0.9, 1.1], for $n=128$, and $k=0.5$. Chaotic pendula are at sites 5, 26, 112. The pattern is plotted for 75 periods ($j=0-75$). (b) Uniform disorder of the pendula lengths in the interval [0.9, 1.1], but excluding the interval [0.998, 1.002] for $n=128$, and $k=0.5$. The pattern is plotted for 75 periods.

coupling strength we use the measure $\sigma(jT)$ to obtain a convenient representation of the bifurcation structure. The results are shown in Fig. 4(a) in which we plot $\sigma(jT)$ as a function of l_{imp} for $k=0.5$, and $N=32$. For each length of the impurity the calculation was carried out for a sufficient number of periods to eliminate transients and the next 16 values of $\sigma(jT)$ were used as a representation of the attractor. Thus $P16$ attractors or attractors of higher periodicity are not recognized as such but rather as chaotic attractors. Nevertheless, Fig. 4(a) shows the existence of a $P1$ attractor coexisting with a $P5$ attractor for impurity lengths up to $l_{\mathrm{imp}}=0.6$. We were able to find more than one $P5$ attractor, however, for clarity only one is included in the bifurcation diagram. The $P1$ attractor is stable from very low values below 0.2 and as high as 0.83. Notice that this spans a large window in which the isolated impurity is in itself chaotic.

Figure 4(b) shows the bifurcation diagram as a function of the impurity length for the same conditions as before but now the coupling constant has been increased to 5.0. In this case the $P1$ behavior is predominant with a small chaotic region around $l_{\mathrm{imp}}\sim 1.0$, which develops through a very clear period doubling sequence. On the right hand side there is a small portion of what seems to be two $P2$ attractors, which quickly bifurcate as the length of the pendulum is decreased. It appears that all the higher order coexisting attractors are associated with edge effects and that there could be a multiplicity of them. Indeed as the number of pendula increases, calculations show that edge effects involve only as few as a couple of extreme pendula.

Finally we examined the effect of disorder and in particular the possibility of obtaining self-organization or frequency locking. We introduced disorder in the lengths of the pendula but restricted the range to be in the region in which the individual pendulum was chaotic, i.e., in the interval $l=[0.998,1.002]$ for a chain of 128 pendula. We found that the emerging pattern was always chaotic for a large number of different realizations of the chain itself and of the initial conditions. On the other hand, if we introduced a uniform and symmetric disorder around $l=1.0$ of 10% so that the pendula lengths are uniformly distributed in the interval [0.9, 1, 1], self-organization was possible. Figure 5(a) shows one such realization for $N=128$ pendula, three of which at sites 5, 26, 112 are chaotic. Except for a few pendula at the edges that execute $P2$ the rest of the pattern is a $P1$ pattern after the rather long transients have settled down. This is not really surprising in view of our earlier results. If all of the pendula were uniformly distributed in the same interval excluding the chaotic range, i.e., excluding the interval [0.998, 1.002] then the resulting pattern always exhibited spatiotemporal organization. Figure 5(b) shows such a realization and one can see clusters of $P1$, $P2$, and $P10$ corresponding to the period of the pattern repetition.

It is also noted that longer chains can be easily synchronized by including more than one impurity at selected sites and appropriate pendulum lengths. The temporal development remains $P1$, however, the spatial pattern adjusts to reflect the new impurity. The particular spatial pattern produced by the introduction of more than one impurity has not been investigated extensively and at this time it still remains a problem for future studies.

IV. SUMMARY

In conclusion we have demonstrated that a chain of chaotic pendula can be frequency locked into a spatiotemporal pattern by introducing an appropriate impurity at a site in the lattice. In most cases a single impurity can tame chaos. We found that a single nonchaotic impurity with an appropriate pendulum length can control and self-organize the dynamics of a chain as long as 128, the limit to which our calculations were performed. Increasing the coupling constant it increases the domain of lengths of the impurity pendulum for which self-organization can be observed. Excluding edge effects, we found that the pattern has always a repetition period equal to the period of the driving signal. Our results suggest that in coupled systems with one positive Lyapunov exponent the introduction of a single impurity can alter the dynamics drastically.

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